# A Recurrence for Counting Graphical Partitions

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**Abstract**

In this paper, we give a recurrence to enumerate the set $G(n)$ of partitions of a positive even integer $n$ which are the degree sequences of simple graphs. The recurrence gives rise to an algorithm to compute the number of elements of $G(n)$ in time $O(n^4)$ using space $O(n^3)$. This appears to be the first method for computing $|G(n)|$ in time bounded by a polynomial in $n$, and it has enabled us to tabulate $|G(n)|$ for even $n \leq 220$.

## 1 Introduction

A *partition* of a positive integer $n$ is a sequence of positive integers $(\pi_1, \pi_2, \ldots, \pi_l)$ satisfying $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_l$ and $\pi_1 + \pi_2 + \ldots + \pi_l = n$. Let $P(n)$ denote the set of all partitions of $n$. $P(0)$ contains only the empty partition, $\lambda$. A partition $\pi \in P(n)$ is *graphical* if it is the degree sequence of some simple undirected graph. For example, $(5, 4, 4, 3, 3, 3, 1)$ is the degree sequence of the graph in Figure 1(a), but $(5, 4, 4, 2, 2, 1)$ is not graphical. Clearly, graphical partitions exist only when $n$ is even, since the sum of the degrees of the vertices of a graph is equal to twice the number of edges. Let $G(n)$ denote the set of graphical partitions of $n$. For convenience, we will call the empty partition graphical, so that $|G(0)| = 1$.

Several necessary and sufficient conditions to determine whether an integer sequence is graphical are surveyed in [SH]. Perhaps the best known is the following condition due to Erdös and Gallai [EG]:

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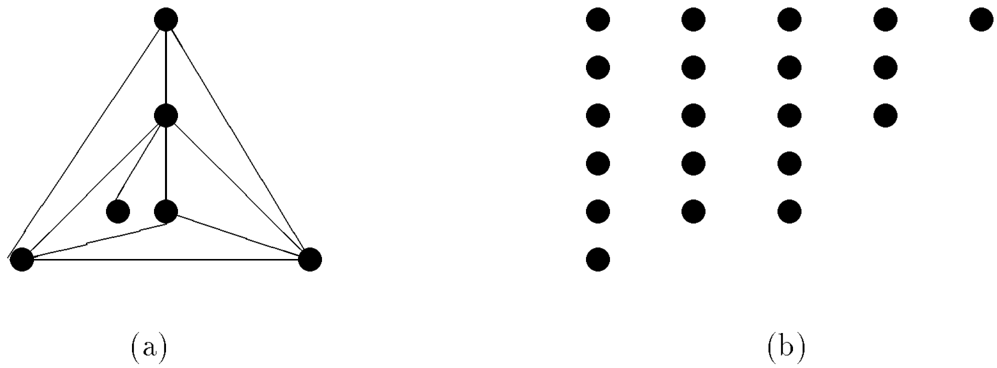


Figure 1: (a) A graph with degree sequence $\pi = (5,4,4,3,3,1)$ and (b) the Ferrars graph of $\pi$.

**[Erdös - Gallai]** A positive integer sequence $(\pi_1, \pi_2, \ldots, \pi_l)$, with $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_l$, is graphical if and only if $\pi_1 + \pi_2 + \ldots + \pi_l$ is even and for $1 \leq j \leq l$,

$$\sum_{i=1}^{j} \pi_i \;\leq\; j(j-1) + \sum_{i=j+1}^{l} \min\{j, \pi_i\}.$$

In Section 2, we use a lesser-known condition to devise a recurrence to enumerate $G(n)$. As shown in Section 3, it can be used to count $G(n)$ in time $O(n^4)$ using space $O(n^3)$.

Our work was motivated by the following question, originally posed by Herbert Wilf, which remains open:

**[Question]** What fraction of the elements of $P(n)$ are graphic? In particular, does the ratio $|G(n)|/|P(n)|$ approach 0 as $n$ approaches infinity?

To even plot the ratio $|G(n)|/|P(n)|$, it is necessary to compute $|G(n)|$, which, in our initial attempts, became a computational burden well before $n = 100$. Using an earlier version of the recurrence, we were able to compute $|G(n)|$ up to $n = 220$. These results are tabulated in Section 4. Where sufficient memory is available, computing $|G(n)|$ up to $n = 1000$ should be feasible.

For a related counting problem, we note that Stanley [St] has obtained a generating function for $f(n)$, the number of sequences $(x_1, x_2, \ldots, x_n)$ which are degree sequences of simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$. Here, $x_i$ is the degree of vertex $v_i$ and the degree sequence is not necessarily nonincreasing.

## 2 The Recurrence

For a partition $\pi = (\pi_1, \ldots, \pi_l)$, the associated *Ferrars graph* is an array of $l$ rows of dots, where row $i$ has $\pi_i$ dots and rows are left justified (Figure 1(b).) Let $\pi'$ denote the conjugate partition $\pi' = (\pi'_1, \ldots, \pi'_m)$ where $m = \pi_1$ and $\pi'_i$ is the number of dots in the $i$-th column of the Ferrars graph of $\pi$. The *Durfee square* of $\pi$ is the largest square subarray of dots in the Ferrars graph of $\pi$. Let $d(\pi)$ denote the size (number of rows) of the Durfee square of $\pi$. The sequence

$$(\pi_1 - \pi'_1, \quad \pi_2 - \pi'_2, \ \ldots, \ \pi_{d(\pi)} - \pi'_{d(\pi)})$$

is the sequence of *successive ranks* of $\pi$ [At]. It will be convenient to work with the negatives of the ranks, so, for $1 \le i \le d(\pi)$, let $r_i(\pi) = \pi'_i - \pi_i$. We call $(r_1(\pi), \ldots r_{d(\pi)}(\pi))$ the sequence of successive *antiranks* of $\pi$.

The necessary and sufficient condition below, attributed to Nash-Williams, is proved in [RA2] and [SH].

**[Nash-Williams]** A partition $\pi$ of an even integer is graphical if and only if for $1 \le j \le d(\pi)$,

$$\sum_{i=1}^{j} r_i(\pi) \ge j.$$

(This condition is called the *Hässelbarth Criterion* by the authors of [SH] since they first saw it in [Has], where it appeared without proof.) It can be shown that for $1 \le j \le d(\pi)$, the $j$-th Nash-Williams condition is equivalent to the $j$-th Erdös-Gallai condition. Furthermore, if conditions $1, 2, \ldots, d(\pi)$ of Erdös-Gallai are satisfied, then so are the remaining Erdös-Gallai conditions [RA2].

Let $P(n, k, l)$ be the set of partitions of $n$ into at most $l$ parts with largest part at most $k$ and define $G(n, k, l)$ to be the set of graphical partitions in $P(n, k, l)$. Let $\pi \in P(n, k, l)$ and let $\alpha$ be obtained from $\pi$ by deleting the first row and column of the Ferrars graph of $\pi$. Then $d(\alpha) = d(\pi) - 1$. Let $s = r_1(\pi) = \pi'_1 - \pi_1$. By the Nash-Williams condition, $\pi$ is graphical if and only if $s > 0$ and for $1 \le j \le d(\alpha)$, the antiranks of $\alpha$ satisfy an $s$-variant of the Nash-Williams conditions:

$$s \ + \ \sum_{i=1}^{j} r_i(\alpha) \ge j.$$

With this in mind, define $P(n,k,l,s)$ for $s \geq 0$ by

$$P(n,k,l,s) = \{ \ \pi \in P(n,k,l) \ \mid \ s \ + \ \sum_{i=1}^{j} r_i(\pi) \ \geq \ j \ \text{ for } \ 1 \leq j \leq d(\pi)\}.$$

Let $P(n,k,l,s) = \emptyset$ if $s < 0$ and note that for $s \geq 0$, $P(n,k,l,s) = \{\lambda\}$ if $P(n,k,l) = \{\lambda\}$.

Lemma 1 below is a restatement of the Nash-Williams condition and Lemma 2 follows since $G(n) = G(n,n,n)$.

**Lemma 1** *For even $n \geq 0$, $G(n,k,l) = P(n,k,l,0)$.*

**Lemma 2** *For even $n \geq 0$, $G(n) = P(n,n,n,0)$.*

Thus, we can compute $|G(n)|$ by computing $|P(n,k,l,s)|$ for appropriate values of the arguments. To this end, let $P'(n,k,l)$ and $P'(n,k,l,s)$, be the subsets of $P(n,k,l)$ and $P(n,k,l,s)$, respectively, consisting of those partitions into *exactly* $l$ parts with largest part of size *exactly* $k$.

**Lemma 3** *For $n > 0$ and $1 \leq k,l,s \leq n$,*

$$|P(n,k,l,s)| - |P'(n,k,l,s)| \ = \ |P(n,k-1,l,s)| + |P(n,k,l-1,s)| - |P(n,k-1,l-1,s)|.$$

**Proof.** ¿From the definitions of $P$ and $P'$ we have

$$P(n,k,l,s) \ \setminus \ P'(n,k,l,s) \ = \ P(n,k-1,l,s) \ \cup \ P(n,k,l-1,s).$$

The set on the left-hand side of this equality has size

$$|P(n,k,l,s)| \ - \ |P'(n,k,l,s)|$$

and by inclusion-exclusion, the set on the right-hand side of the equality has size

$$|P(n,k-1,l,s)| \ + \ |P(n,k,l-1,s)| \ - \ |P(n,k-1,l,s) \ \cap \ P(n,k,l-1,s)|.$$

The result follows since the intersection in the last term is $P(n,k-1,l-1,s)$. $\Box$

**Lemma 4** *Assume $n > 0$, $1 \leq k,l, \leq n$, and $s \geq 0$. Then*

$$|P'(n,k,l,s)| \ = \ |P(n-k-l+1, \quad k-1, \quad l-1, \quad s+l-k-1)|.$$

**Proof.** Define a function $f$ on $P'(n,k,l)$ by $f(\pi) = \alpha$, where $\alpha$ is obtained from $\pi$ by deleting the first row and column in the Ferrars graph of $\pi$. Given the assumptions of the theorem, if $P'(n,k,l) = \emptyset$ then either (1) $n < k+l-1$, in which case $n-k-l+1 < 0$ or (2) $n > kl$, which implies $n-k-l+1 > kl-k-l+1 = (k-1)(l-1)$. In either of these cases, $P(n-k-l+1, k-1, l-1) = \emptyset$. If $P'(n,k,l)$ contains only the partition $(k,1,\ldots,1)$, then $f((k,1,\ldots,1)) = \lambda$, $n-k-l+1 = 0$, and $P(n-k-l+1, k-1, l-1) = \{\lambda\}$. Otherwise, $d(\alpha) = d(\pi) - 1$ and $\alpha = (\alpha_1, \ldots \alpha_m)$ where $m = \pi'_2 - 1$, $\alpha_i = \pi_{i+1} - 1$ for $1 \leq i \leq m$ and $\alpha'_i = \pi'_{i+1} - 1$ for $1 \leq i \leq d(\pi) - 1$. Clearly, $f$ is a bijection between $P'(n,k,l)$ and $P(n-k-l+1,\ k-1,\ l-1)$ Furthermore, for $1 \leq j \leq d(\pi)$,

$$\begin{aligned} s + \sum_{i=1}^{j}(\pi'_i - \pi_i) &= (s+l-k) + \sum_{i=2}^{j}(\pi'_i - \pi_i) \\ &= (s+l-k) + \sum_{i=2}^{j}((\pi'_i - 1) - (\pi_i - 1)) \\ &= (s+l-k) + \sum_{i=1}^{j-1}(\alpha'_i - \alpha_i). \end{aligned}$$

Thus

$$s \ + \ \sum_{i=1}^{j} r_i(\pi) \ \geq \ j \iff$$

$$(s+l-k-1) \ + \ \sum_{i=1}^{j-1} r_i(\alpha) \ \geq \ j-1.$$

This establishes that $\pi \in P'(n,k,l,s) \iff \alpha \in P(n-k-l+1,\ k-1,\ l-1,\ s+l-k-1)$. $\square$

**Lemma 5** *$P(n,k,l) = P(n,k,l,n) = P(n,k,l,s)$ for $s \geq n$.*

**Proof.** Note that for any $\pi \in P(n,k,l)$ and $1 \leq j \leq \pi(d)$,

$$\sum_{i=1}^{j}(\pi'_i - \pi_i - 1) \ \geq \ \sum_{i=1}^{j} -\pi_i \ \geq \ -n.$$

Thus, $\pi \in P(n,k,l,n)$, which means $P(n,k,l) \subseteq P(n,k,l,n)$. By definition, for $t' \geq t \geq 0$, $P(n,k,l,t) \subseteq P(n,k,l,t')$, thus for any $s \geq n$, $P(n,k,l,n) \subseteq P(n,k,l,s)$. The result follows since $P(n,k,l,s) \subseteq P(n,k,l)$. $\square$

The resulting recurrence for counting $|P(n,k,l,s)|$ is given in the following.

**Theorem 1** *$|P(n,k,l,s)|$ is defined by:*

$|P(n,k,l,s)| =$

| | | | |
|---|---|---|---|
| *if* $((n<0)$ *or* $(k<0)$ *or* $(l<0)$ *or* $(s<0))$ | *then :* | *0* | (1) |
| *else if* $n=0$ | *then:* | *1* | (2) |
| *else if* $(k=0)$ *or* $(l=0)$ | *then:* | *0* | (3) |
| *else if* $(k>n)$ | *then:* | $\lvert P(n,n,l,s)\rvert$ | (4) |
| *else if* $(l>n)$ | *then:* | $\lvert P(n,k,n,s)\rvert$ | (5) |
| *else if* $(s>n)$ | *then:* | $\lvert P(n,k,l,n)\rvert$ | (6) |
| *else:* $\lvert P(n,k-1,l,s)\rvert + \lvert P(n,k,l-1,s)\rvert - \lvert P(n,k-1,l-1,s)\rvert$ $+\lvert P(n-k-l+1,\ k-1,\ l-1,\ s+l-k-1)\rvert$ | | | (7) |

*Proof.* $P(n,k,l,s)$ was defined to be empty when $s<0$. For the remaining conditions in (1) through (5) the value of $|P(n,k,l,s)|$ is clear. Condition (6) follows from Lemma 5. For the general case (7), equate $|P'(n,k,l,s)|$ in Lemmas 3 and 4 and then solve for $|P(n,k,l,s)|$. □

## 3 The Algorithm

The recurrence of Theorem 1 for computing $|P(n,k,l,s)|$ has a straightforward implementation as a dynamic programming algorithm which fills a 4-dimensional table of entries $T[a,b,c,d] = |P(a,b,c,d)|$ where $0 \le a \le n$, $0 \le b \le k$, $0 \le c \le l$, and $0 \le d \le n$. The table is filled in any order which guarantees that when the time comes to fill entry $T[n',k',l',s']$, the required entries $T[n',k'-1,l',s']$, $T[n',k',l'-1,s']$, $T[n',k'-1,l'-1,s']$, and $T[n'-k'-l'+1,k'-1,l'-1,s'+l'-k'-1]$ have already been filled and can be read from the table. The table uses space $O(n^2kl)$ and only constant time is required to fill in each entry. In particular, computing $|G(n)| = |P(n,n,n,0)|$ takes time and space $O(n^4)$.

The space can be asymptotically improved as follows. For $0 \le c \le l$, let $T_c$ be the 3-dimensional table of entries $T_c[a,b,d] = |P(a,b,c,d)|$ for $0 \le a \le n$, $0 \le b \le k$, and $0 \le d \le n$. Then $|P(n,k,l,s)|$ can be computed by computing successively the tables $T_0$, $T_1, \ldots T_l$. Note from the recurrence of Theorem 1 that computing entries in table $T_c$ requires access only to values in table $T_c$ or table $T_{c-1}$. Thus, in computing $|P(n,k,l,s)|$, no more than two 3-dimensional tables need to be stored at any given time, reducing the space required to $O(n^2k)$. Thus, computing $|G(n)|$ can be done in $O(n^4)$ time with $O(n^3)$ space.

## 4 Concluding Remarks

Even with this polynomial time algorithm, computing $|G(n)|$ for $n > 200$ quickly becomes impractical because of the huge space requirements. An additional burden on space is that $|G(n)|$ gets large quickly so that some method must be used to manipulate and allocate enough storage for these large numbers. The following strategy was suggested by the referee: Select small primes $p_1 < p_2 < \ldots < p_s$ so that $p_1 p_2 \ldots p_s > G(n)$. For $i = 1, \ldots, s$, use the recurrence of Theorem 1 to compute $G_i(n) = G(n) \bmod (p_i)$. Then by the Chinese Remainder Theorem, $G(n)$ can be recovered from $G_1(n), \ldots, G_s(n)$. If, for example, the primes can be represented with 8 bits, time $O(n^4)$ will be spent computing each of $s$ tables, but the 3-dimensional tables now need store only 8-bit integers.

For those interested in the values $|G(n)|$, or in the ratio $|G(n)|/|P(n)|$ from the open question of Section 1, we include Tables 1 and 2. To the best of our knowledge, the values had previously been computed only through $n = 40$, as noted in [ER] in an acknowledgement to Ron Read. ¿From the data, it seems reasonable to make the conjecture that for even $n \geq 18$, $|G(n)|/|P(n)|$ is monotone decreasing, but we are not aware of any proof of this. The best results known at this time are that

$$\overline{\lim}_{n\to\infty} \frac{\sqrt{n}\ |G(n)|}{|P(n)|} \ \geq \ \frac{\pi}{\sqrt{6}}$$

(so the ratio cannot go to 0 faster than $1/\sqrt{n}$ [ER]) and that [RA1]

$$\overline{\lim}_{n\to\infty} \frac{|G(n)|}{|P(n)|} \ \leq \ .25.$$

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| $n$ | $\|G(n)\|$ | $\|P(n)\|$ | $\|G(n)\|/\|P(n)\|$ |
|---|---|---|---|
| 2 | 1 | 2 | 0.500000 |
| 4 | 2 | 5 | 0.400000 |
| 6 | 5 | 11 | 0.454545 |
| 8 | 9 | 22 | 0.409091 |
| 10 | 17 | 42 | 0.404762 |
| 12 | 31 | 77 | 0.402597 |
| 14 | 54 | 135 | 0.400000 |
| 16 | 90 | 231 | 0.389610 |
| 18 | 151 | 385 | 0.392208 |
| 20 | 244 | 627 | 0.389155 |
| 22 | 387 | 1002 | 0.386228 |
| 24 | 607 | 1575 | 0.385397 |
| 26 | 933 | 2436 | 0.383005 |
| 28 | 1420 | 3718 | 0.381926 |
| 30 | 2136 | 5604 | 0.381156 |
| 32 | 3173 | 8349 | 0.380046 |
| 34 | 4657 | 12310 | 0.378310 |
| 36 | 6799 | 17977 | 0.378205 |
| 38 | 9803 | 26015 | 0.376821 |
| 40 | 14048 | 37338 | 0.376239 |
| 42 | 19956 | 53174 | 0.375296 |
| 44 | 28179 | 75175 | 0.374845 |
| 46 | 39467 | 105558 | 0.373889 |
| 48 | 54996 | 147273 | 0.373429 |
| 50 | 76104 | 204226 | 0.372646 |
| 52 | 104802 | 281589 | 0.372181 |
| 54 | 143481 | 386155 | 0.371563 |
| 56 | 195485 | 526823 | 0.371064 |
| 58 | 264941 | 715220 | 0.370433 |
| 60 | 357635 | 966467 | 0.370044 |
| 62 | 480408 | 1300156 | 0.369500 |
| 64 | 642723 | 1741630 | 0.369035 |
| 66 | 856398 | 2323520 | 0.368578 |
| 68 | 1136715 | 3087735 | 0.368139 |
| 70 | 1503172 | 4087968 | 0.367706 |
| 72 | 1980785 | 5392783 | 0.367303 |
| 74 | 2601057 | 7089500 | 0.366889 |
| 76 | 3404301 | 9289091 | 0.366484 |
| 78 | 4441779 | 12132164 | 0.366116 |
| 80 | 5777292 | 15796476 | 0.365733 |
| 82 | 7492373 | 20506255 | 0.365370 |
| 84 | 9688780 | 26543660 | 0.365013 |
| 86 | 12494653 | 34262962 | 0.364669 |
| 88 | 16069159 | 44108109 | 0.364313 |
| 90 | 20614755 | 56634173 | 0.363999 |
| 92 | 26377657 | 72533807 | 0.363660 |
| 94 | 33671320 | 92669720 | 0.363348 |
| 96 | 42878858 | 118114304 | 0.363028 |
| 98 | 54481054 | 150198136 | 0.362728 |
| 100 | 69065657 | 190569292 | 0.362418 |
| 102 | 87370195 | 241265379 | 0.362133 |
| 104 | 110287904 | 304801365 | 0.361835 |
| 106 | 138937246 | 384276336 | 0.361556 |
| 108 | 174675809 | 483502844 | 0.361272 |
| 110 | 219186741 | 607163746 | 0.361001 |

Table 1: Sizes of $G(n)$ and $P(n)$ and their ratio for $2 \leq n \leq 110$.

| $n$ | $\|G(n)\|$ | $\|P(n)\|$ | $\|G(n)\|/\|P(n)\|$ |
|---|---|---|---|
| 112 | 274512656 | 761002156 | 0.360725 |
| 114 | 343181668 | 952050665 | 0.360466 |
| 116 | 428244215 | 1188908248 | 0.360200 |
| 118 | 533464959 | 1482074143 | 0.359945 |
| 120 | 663394137 | 1844349560 | 0.359690 |
| 122 | 823598382 | 2291320912 | 0.359443 |
| 124 | 1020807584 | 2841940500 | 0.359194 |
| 126 | 1263243192 | 3519222692 | 0.358955 |
| 128 | 1560795436 | 4351078600 | 0.358715 |
| 130 | 1925513465 | 5371315400 | 0.358481 |
| 132 | 2371901882 | 6620830889 | 0.358248 |
| 134 | 2917523822 | 8149040695 | 0.358021 |
| 136 | 3583515700 | 10015581680 | 0.357794 |
| 138 | 4395408234 | 12292341831 | 0.357573 |
| 140 | 5383833857 | 15065878135 | 0.357353 |
| 142 | 6585699894 | 18440293320 | 0.357136 |
| 144 | 8045274746 | 22540654445 | 0.356923 |
| 146 | 9815656018 | 27517052599 | 0.356711 |
| 148 | 11960467332 | 33549419497 | 0.356503 |
| 150 | 14555902348 | 40853235313 | 0.356297 |
| 152 | 17692990183 | 49686288421 | 0.356094 |
| 154 | 21480510518 | 60356673280 | 0.355893 |
| 156 | 26048320019 | 73232243759 | 0.355695 |
| 158 | 31551087790 | 88751778802 | 0.355498 |
| 160 | 38173235010 | 107438159466 | 0.355304 |
| 162 | 46134037871 | 129913904637 | 0.355112 |
| 164 | 55694314567 | 156919475295 | 0.354923 |
| 166 | 67163674478 | 189334822579 | 0.354735 |
| 168 | 80909973315 | 228204732751 | 0.354550 |
| 170 | 97368672089 | 274768617130 | 0.354366 |
| 172 | 117056456152 | 330495499613 | 0.354185 |
| 174 | 140584220188 | 397125074750 | 0.354005 |
| 176 | 168675124141 | 476715857290 | 0.353827 |
| 178 | 202182888436 | 571701605655 | 0.353651 |
| 180 | 242116891036 | 684957390936 | 0.353477 |
| 182 | 289666252014 | 819876908323 | 0.353305 |
| 184 | 346234896845 | 980462880430 | 0.353134 |
| 186 | 413474657328 | 1171432692373 | 0.352965 |
| 188 | 493331835384 | 1398341745571 | 0.352700 |
| 190 | 588093594457 | 1667727404093 | 0.352632 |
| 192 | 700451190712 | 1987276856363 | 0.352468 |
| 194 | 833561537987 | 2366022741845 | 0.352305 |
| 196 | 991134281267 | 2814570987591 | 0.352144 |
| 198 | 1177516049387 | 3345365983698 | 0.351984 |
| 200 | 1397805210533 | 3972999029388 | 0.351826 |
| 202 | 1657968320899 | 4714566886083 | 0.351669 |
| 204 | 1964994991232 | 5590088317495 | 0.351514 |
| 206 | 2327052859551 | 6622987708040 | 0.351360 |
| 208 | 2753697110356 | 7840656226137 | 0.351207 |
| 210 | 3256081386335 | 9275102575355 | 0.351056 |
| 212 | 3847232865612 | 10963707205259 | 0.350906 |
| 214 | 4542341563460 | 12950095925895 | 0.350757 |
| 216 | 5359127512113 | 15285151248481 | 0.350610 |
| 218 | 6318223879596 | 18028182516671 | 0.350464 |
| 220 | 7443670977177 | 21248279009367 | 0.350319 |

Table 2: Sizes of $G(n)$ and $P(n)$ and their ratio for $112 \leq n \leq 220$.

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